

THE INITIAL-BOUNDARY VALUE PROBLEM FOR THE COMPRESSIBLE VISCOELASTIC FLUIDS

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ABSTRACT. The global existence of strong solution to the initial-boundary value problem of the three-dimensional compressible viscoelastic fluids near equilibrium is established in a bounded domain. Uniform estimates in $W^{1,q}$ with $q > 3$ on the density and deformation gradient are also obtained. All the results apply to the two-dimensional case.

1. INTRODUCTION

Elastic solids and viscous fluids are two extremes of material behavior. Viscoelastic fluids show intermediate behavior with some remarkable phenomena due to their “elastic” nature. These fluids exhibit a combination of both fluid and solid characteristics, keep memory of their past deformations, and their behaviour is a function of these old deformations. Viscoelastic fluids have a wide range of applications and hence have received a great deal of interest. Examples and applications of viscoelastic fluids include from oil, liquid polymers, mucus, liquid soap, toothpaste, clay, ceramics, gels, some types of suspensions, to bioactive fluids, coatings and drug delivery systems for controlled drug release, scaffolds for tissue engineering, and viscoelastic blood flow past valves; see [8] for more applications. For the viscoelastic materials, the competition between the kinetic energy and the internal elastic energy through the special transport properties of their respective internal elastic variables makes the materials more untractable in understanding their behavior, since any distortion of microstructures, patterns or configurations in the dynamical flow will involve the deformation tensor. For classical simple fluids, the internal energy can be determined solely by the determinant of the deformation tensor; however, the internal energy of complex fluids carries all the information of the deformation tensor. The interaction between the microscopic elastic properties and the macroscopic fluid motions leads to the rich and complicated rheological phenomena in viscoelastic fluids, and also causes formidable analytic and numerical challenges in mathematical analysis. The equations of the compressible viscoelastic fluids of Oldroyd type ([21, 22]) in three spatial dimensions take the following form [7, 17, 24]:

$$\varrho_t + \operatorname{div}(\varrho \mathbf{u}) = 0, \quad (1.1a)$$

$$(\varrho \mathbf{u})_t + \operatorname{div}(\varrho \mathbf{u} \otimes \mathbf{u}) - \mu \Delta \mathbf{u} - (\mu + \lambda) \nabla \operatorname{div} \mathbf{u} + \nabla P(\varrho) = \operatorname{div}(\varrho \mathbf{F} \mathbf{F}^\top), \quad (1.1b)$$

$$\mathbf{F}_t + \mathbf{u} \cdot \nabla \mathbf{F} = \nabla \mathbf{u} \mathbf{F}, \quad (1.1c)$$

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where ϱ stands for the density, $\mathbf{u} \in \mathbb{R}^3$ the velocity, and $\mathbf{F} \in M^{3 \times 3}$ (the set of 3×3 matrices) the deformation gradient. The viscosity coefficients $\mu > 0$ and λ satisfy

$$\mu > 0, \quad \text{and} \quad 2\mu + 3\lambda > 0,$$

which ensures that the operator $-\mu\Delta\mathbf{u} - (\mu + \lambda)\nabla\text{div}\mathbf{u}$ is a strongly elliptic operator. The term $P(\varrho)$ represents the pressure for the barotropic case and is an increasing and convex smooth function of $\varrho > 0$ with $P(1) > 0$. The symbol \otimes denotes the Kronecker tensor product and \mathbf{F}^\top means the transpose matrix of \mathbf{F} . As usual we call equation (1.1a) the continuity equation. For system (1.1), the corresponding elastic energy is chosen to be the special form of the Hookean linear elasticity:

$$W(\mathbf{F}) = \frac{1}{2}|\mathbf{F}|^2,$$

which, however, does not reduce the essential difficulties for analysis. The methods and results of this paper can be applied to more general cases.

In this paper, we consider equations (1.1) in the three-dimensional bounded domain $\Omega \subset \mathbb{R}^3$ with sufficiently smooth boundary, subject to the following initial and boundary conditions:

$$\begin{cases} (\varrho, \mathbf{u}, \mathbf{F})|_{t=0} = (\varrho_0(x), \mathbf{u}_0(x), \mathbf{F}_0(x)), & x \in \Omega, \\ \mathbf{u}|_{\partial\Omega} = 0, \end{cases} \quad (1.2)$$

and we are interested in the existence and uniqueness of strong solution to the initial-boundary value problem (1.1)-(1.2) near its equilibrium state. Here the equilibrium state of the system (1.1) is defined as: ϱ is a positive constant (for simplicity, $\varrho = 1$), $\mathbf{u} = 0$, and $\mathbf{F} = I$ (the identity matrix in $M^{3 \times 3}$). We denote the perturbations of the density, the velocity, and the deformation gradient about the equilibrium by ρ , \mathbf{u} , and E , respectively, that is,

$$\varrho = 1 + \rho, \quad \mathbf{F} = I + E.$$

Then, system (1.1) becomes equivalently

$$\rho_t + \text{div}(\rho\mathbf{u}) + \text{div}\mathbf{u} = 0, \quad (1.3a)$$

$$\begin{aligned} ((1 + \rho)\mathbf{u})_t + \text{div}((1 + \rho)\mathbf{u} \otimes \mathbf{u}) - \mu\Delta\mathbf{u} - (\mu + \lambda)\nabla\text{div}\mathbf{u} + \nabla P(1 + \rho) \\ = \text{div}((1 + \rho)(I + E)(I + E)^\top), \end{aligned} \quad (1.3b)$$

$$E_t + \mathbf{u} \cdot \nabla E = \nabla\mathbf{u}E + \nabla\mathbf{u}, \quad (1.3c)$$

with the initial and boundary conditions:

$$\begin{cases} (\rho, \mathbf{u}, E)|_{t=0} = (\rho_0(x), \mathbf{u}_0(x), E_0(x)) = (\varrho_0(x) - 1, \mathbf{u}_0(x), \mathbf{F}_0(x) - I), & x \in \Omega, \\ \mathbf{u}|_{\partial\Omega} = 0. \end{cases} \quad (1.4)$$

By a *strong solution*, we mean a triplet (ρ, \mathbf{u}, E) satisfying (1.3) almost everywhere with the initial and boundary conditions (1.4), in particular, $(\rho, \mathbf{u}, E)(\cdot, t) \in W^{1,q} \times (W^{2,q})^3 \times (W^{1,q})^{3 \times 3}$, $q \in (3, \infty)$ in this paper.

When the density is a constant, system (1.1) governs the homogeneous incompressible viscoelastic fluids, and there exist rich results in the literature for the global existence of classical solutions (namely in H^3 or other functional spaces with much higher regularity); see [1, 2, 9, 10, 11, 12, 13, 14, 17] and the references therein. When the density is not a constant, the question related to existence becomes much more complicated. In [6] the

authors considered the global existence of classical solutions of small perturbation near its equilibrium for the compressible viscoelastic fluids in critical spaces (a functional space in which the system is scaling invariant), see also [23]. For the local existence of strong solutions with large initial data, see [5]. One of the main difficulties in proving the global existence for compressible viscoelastic fluids is the lacking of the dissipative estimate for the deformation gradient. To overcome this difficulty, the authors in [10] introduced an auxiliary function to obtain the dissipative estimate, while the authors in [12] directly deal with the quantities such as $\Delta \mathbf{u} + \operatorname{div} \mathbf{F}$. Those methods can provide them with some good estimates, partly because of their high regularity of (\mathbf{u}, \mathbf{F}) . However, in this paper, we deal with the strong solution with much less regularity for $(\mathbf{u}, \mathbf{F}) \in W^{2,q} \times W^{1,q}$, $q \in (3, \infty)$, hence those methods do not apply (more precisely, we can not apply the standard energy method in our situation). For our purpose, we first obtain a uniform estimate for the linearized momentum equation using the maximal regularities of the Stokes equations and parabolic equations, and then we find that a combination between the velocity and the convolution of the divergence of the deformation gradient with the fundamental solution of Laplace operator will develop some good dissipative estimates which are very useful for the global existence.

When the initial-boundary value problem (1.3)-(1.4) is considered, several difficulties need to be addressed:

- (1) For the initial-boundary value problem, the boundary condition can not be prescribed as zero for the density or the deformation gradient, and hence taking the derivatives of equations up to arbitrary orders and integrating by parts can not be applied. To overcome this difficulty, we need to establish a uniform estimate for the linearized momentum equation with the help of the standard $L^p - L^q$ estimates for the Stokes equations and the variants for the heat equation;
- (2) The functional framework now is the general L^q space with $q > 3$. This setting obviously exclude the classical methods which are the key tools in [5, 10, 13, 14, 23]. To handle this difficulty, the classical Poincaré's inequality is used and a new conserved quantity for (1.1) will be needed.

One key observation of this work is that under the condition (2.4) (see Section 2) on the curl of the deformation gradient initially, not only the curl of the deformation gradient at any positive time is a higher order term, but also is the sum of the gradient of density and the divergence of the deformation gradient (see (5.4)), which motivates us to use the $L^p - L^q$ estimate of Stokes equations. Furthermore, the divergence of the difference between the deformation gradient and its transpose is also a higher order term, see (5.24).

The viscoelastic fluid system (1.1) can be regarded as a combination of the inhomogeneous compressible Navier-Stokes equations with the source term $\operatorname{div}(\rho \mathbf{F} \mathbf{F}^\top)$ and the equation (1.1c). For the global existence of classical solutions with small perturbation near an equilibrium for the compressible Navier-Stokes equations, we refer the reader to [18, 19, 20] and the references cited therein. We remark that, for the nonlinear inviscid elastic systems, the existence of solutions was established by Sideris-Thomases in [26] under the null condition; see also [25] for a related discussion.

The existence of global weak solutions with large initial data of (1.1) is still an outstanding open question. In this direction for the homogeneous incompressible viscoelastic fluids, when the contribution of the strain rate (symmetric part of $\nabla \mathbf{u}$) in the constitutive

equation is neglected, Lions-Masmoudi in [16] proved the global existence of weak solutions with large initial data for the Oldroyd model. Also Lin-Liu-Zhang in [13] proved the existence of global weak solutions with large initial data for the incompressible viscoelastic fluids when the velocity satisfies the Lipschitz condition. When dealing with the global existence of weak solutions of the viscoelastic fluid system (1.1) with large data, the rapid oscillation of the density and the non-compatibility between the quadratic form and the weak convergence are two of the major difficulties.

The rest of the paper is organized as follows. In Section 2, we recall several intrinsic properties of the system (1.1) and also show a new conserved quantity. In Section 3, we introduce the functional spaces and state our main results, including the local existence and uniqueness of the strong solution to the system (1.3)-(1.4), as well as the global existence. In Section 4, we prove the uniform estimate for the linearized momentum equation. In Section 5, the main goal is to obtain a series of uniform estimates for the gradient of the density and the gradient of the deformation gradient. In Section 6, we establish some uniform in time *a priori* estimates on the dissipation of the deformation gradient and the density, and finally finish the proof of our main theorem.

2. SOME INTRINSIC PROPERTIES OF SYSTEM (1.1)

In this section, we recall some intrinsic properties of system (1.1) and introduce a new conserved quantity. First, for the system (1.1) it was proved that (see Lemmas 6.1 and 6.2 in Hu-Wang [6], and also [23]).

Proposition 2.1. *Assume that $(\varrho, \mathbf{u}, \mathbf{F})$ is a solution of system (1.1). Then the following identities*

$$\varrho \det(\mathbf{F}) = 1, \quad (2.1)$$

$$\operatorname{div}(\varrho \mathbf{F}^\top) = 0, \quad (2.2)$$

and

$$\mathbf{F}_{lk} \nabla_l \mathbf{F}_{ij} = \mathbf{F}_{lj} \nabla_l \mathbf{F}_{ik} \quad (2.3)$$

hold for all time $t > 0$ if they are satisfied initially.

Note that (2.3) can be interpreted in terms of the perturbation about the equilibrium as

$$\partial_{x_k} E_{ij} - \partial_{x_j} E_{ik} = E_{lj} \nabla_l E_{ik} - E_{lk} \nabla_l E_{ij}. \quad (2.4)$$

Similarly, (2.1) can be also interpreted in terms of the perturbation about the equilibrium as

$$(1 + \rho) \det(I + E) = 1,$$

which implies

$$(1 + \rho) \left(1 + \operatorname{tr} E + \frac{1}{2} [(\operatorname{tr} E)^2 - \operatorname{tr}(E^2)] + \det E \right) = 1 \quad (2.5)$$

since for any 3×3 matrix E , we have

$$\det(I + E) = 1 + \operatorname{tr} E + \frac{1}{2} [(\operatorname{tr} E)^2 - \operatorname{tr}(E^2)] + \det E.$$

The identity (2.5) further implies

$$\operatorname{tr} E = -\rho - \rho \operatorname{tr} E + (1 + \rho) \left[\frac{1}{2} [\operatorname{tr}(E^2) - (\operatorname{tr} E)^2] - \det E \right]. \quad (2.6)$$

It is worthy to pointing out that by a similar argument, the constraint in the two-dimensional case between the perturbations of density and the deformation gradient has the form of

$$\operatorname{tr} E = -\rho - \rho \operatorname{tr} E - (1 + \rho) \det E.$$

Similar to the conservation of mass due to the continuity equation (1.1a), the quantity

$$\int_{\Omega} \varrho \mathbf{F}_i dx$$

is also conserved, where \mathbf{F}_i is the i -th column of the matrix \mathbf{F} .

Proposition 2.2. *If $\int_{\Omega} \varrho_0 (\mathbf{F}_0)_i dx = 0$ and $\operatorname{div}(\varrho_0 \mathbf{F}_0^\top) = 0$, then for all time $t > 0$,*

$$\int_{\Omega} \varrho \mathbf{F}_i(x, t) dx = 0. \quad (2.7)$$

Proof. Indeed, from (1.1a) and (1.1c), we deduce that

$$\partial_t(\varrho \mathbf{F}) + \mathbf{u} \cdot \nabla(\varrho \mathbf{F}) = \nabla \mathbf{u}(\varrho \mathbf{F}) - \varrho \mathbf{F} \operatorname{div} \mathbf{u}. \quad (2.8)$$

On the other hand, from the vector identity

$$\operatorname{curl}(A \times B) = A \operatorname{div} B - B \operatorname{div} A + (B \cdot \nabla) A - (A \cdot \nabla) B$$

for all vector-valued functions A , B , and $\operatorname{div}(\varrho \mathbf{F}^\top) = 0$, one has

$$\begin{aligned} & \mathbf{u} \cdot \nabla(\varrho \mathbf{F}_i) - \nabla \mathbf{u} \varrho \mathbf{F}_i + \varrho \mathbf{F}_i \operatorname{div} \mathbf{u} \\ &= \mathbf{u} \cdot \nabla(\varrho \mathbf{F}_i) - \varrho \mathbf{F}_i \cdot \nabla \mathbf{u} + \varrho \mathbf{F}_i \operatorname{div} \mathbf{u} \\ &= -\nabla \times (\mathbf{u} \times \varrho \mathbf{F}_i). \end{aligned}$$

Thus, we can rewrite (2.8) as

$$\partial_t(\varrho \mathbf{F}_i) - \nabla \times (\mathbf{u} \times \varrho \mathbf{F}_i) = 0.$$

Integrating this identity over Ω gives

$$\frac{d}{dt} \int_{\Omega} \varrho \mathbf{F}_i dx = 0. \quad (2.9)$$

The proof is complete. □

3. MAIN RESULTS

In this paper, the standard notations for Sobolev spaces $W^{s,q}$ and Besov spaces B_{pq}^s will be used, and the following interpolation spaces will be needed:

$$X_p^{2(1-\frac{1}{p})} = (L^q(\Omega), W^{2,q}(\Omega))_{1-\frac{1}{p}, p} = B_{qp}^{2(1-\frac{1}{p})},$$

and

$$Y_p^{1-\frac{1}{p}} = (L^q(\Omega), W^{1,q}(\Omega))_{1-\frac{1}{p}, p} = B_{qp}^{1-\frac{1}{p}}.$$

Now we introduce the following functional spaces to which the solution and initial condition of the system (1.3) will belong. Given $1 \leq p, q < \infty$ and $T > 0$, we set $Q_T = \Omega \times (0, T)$, and

$$\mathcal{W}^{p,q}(0, T) := \{\mathbf{u} \in W^{1,p}(0, T; (L^q(\Omega))^3) \cap L^p(0, T; (W^{2,q}(\Omega))^3)\}$$

with the norm

$$\|\mathbf{u}\|_{\mathcal{W}^{p,q}(0, T)} := \|\mathbf{u}_t\|_{L^p(0, T; L^q(\Omega))} + \|\mathbf{u}\|_{L^p(0, T; W^{2,q}(\Omega))},$$

as well as

$$V_0^{p,q} := \left(X_p^{2(1-\frac{1}{p})} \cap Y_p^{1-\frac{1}{p}} \right)^3 \times (W^{1,q}(\Omega))^{10}$$

with the norm

$$\|(f, g)\|_{V_0^{p,q}} := \|f\|_{X_p^{2(1-\frac{1}{p})}} + \|f\|_{Y_p^{1-\frac{1}{p}}} + \|g\|_{W^{1,q}(\Omega)}.$$

For simplicity of notations, we drop the superscripts p, q in $\mathcal{W}^{p,q}$ and $V_0^{p,q}$, that is, we denote

$$\mathcal{W} := \mathcal{W}^{p,q}, \quad V_0 := V_0^{p,q}.$$

For large initial data, the following local in time well-posedness can be obtained as in [5].

Theorem 3.1. *Assume that Ω is a bounded domain in \mathbb{R}^3 with $C^{2+\beta}$ ($\beta > 0$) boundary and $(\mathbf{u}_0, \rho_0, E_0) \in V_0$ with $p \in [2, \infty), q \in (3, \infty)$. There exists a positive constant T_0 such that the initial-boundary value problem (1.3)-(1.4) has a unique strong solution on $\Omega \times (0, T_0)$, satisfying*

$$(\mathbf{u}, \rho, E) \in \mathcal{W}(0, T_0) \times (W^{1,p}(0, T_0; L^q(\Omega)) \cap L^p(0, T_0; W^{1,q}(\Omega)))^{10}.$$

The argument for proving Theorem 3.1 is similar to that in [5], thus we omit the proof here.

Remark 3.1. An interesting case is the case $q \leq p$. Indeed, by the real interpolation method, we have

$$W^{2(1-\frac{1}{p}), q} \subset B_{qp}^{2(1-\frac{1}{p})} = X_p^{2(1-\frac{1}{p})},$$

and

$$W^{1-\frac{1}{p}, q} \subset B_{qp}^{1-\frac{1}{p}} = Z_p^{1-\frac{1}{p}}.$$

Then, if we replace the functional space $V_0^{p,q}$ in Theorem 3.1 by

$$\mathcal{V}_0^{p,q} := \left((W^{2(1-\frac{1}{p}), q}(\Omega))^3 \cap (W^{1-\frac{1}{p}, q}(\Omega))^3 \right) \times (W^{1,q}(\Omega))^{10},$$

Theorem 3.1 is still valid.

Now our main result can be stated as follows.

Theorem 3.2. *Assume that Ω is a bounded domain in \mathbb{R}^3 with $C^{2+\beta}$ ($\beta > 0$) boundary and $(\mathbf{u}_0, \rho_0, E_0) \in V_0$ with $p \in [2, \infty), q \in (3, \infty)$. There exists a positive number $R < 1$ such that if the initial data satisfies (2.2), (2.3), (2.7), and*

$$\|(\mathbf{u}_0, \rho_0, E_0)\|_{V_0} \leq R^2,$$

then the initial-boundary value problem (1.3)-(1.4) has a unique global strong solution

$$(\mathbf{u}, \rho, E) \in \mathcal{W}(0, \infty) \times (W^{1,p}(0, \infty; L^q(\Omega)) \cap L^p(0, \infty; W^{1,q}(\Omega)))^{10},$$

satisfying

$$\begin{cases} \|\mathbf{u}\|_{\mathcal{W}(0, \infty)} < R; \\ \|\rho\|_{L^\infty(0, \infty; W^{1,q}(\Omega))} < R; \\ \|E\|_{L^\infty(0, \infty; W^{1,q}(\Omega))} < R. \end{cases} \quad (3.1)$$

The rest of this paper is devoted to the proof of Theorem 3.2. For simplicity of presentation, we will assume that

$$\begin{aligned} |\Omega| &= 1, \\ \int_{\Omega} \rho_0 dx &= 0, \end{aligned} \quad (3.2)$$

and

$$\int_{\Omega} (1 + \rho_0)(I + E_0)_{ij} dx = \delta_{ij} = \begin{cases} 0, & \text{if } i \neq j; \\ 1, & \text{if } i = j. \end{cases} \quad (3.3)$$

Note that from the continuity equation and (2.9) in Section 2, the identities (3.2) and (3.3) hold also for all positive $t > 0$.

4. ESTIMATES FOR THE LINEARIZED MOMENTUM EQUATION

In this section, we consider the following linearized equation:

$$\begin{cases} \partial_t f - \mu \Delta f - (\mu + \lambda) \nabla \operatorname{div} f + \nabla h = g \\ f(0) = f_0, \quad f|_{\partial\Omega} = 0, \end{cases} \quad (4.1)$$

where $f \in \mathcal{W}(0, T)$, $g \in (L^p(0, T; L^q(\Omega)))^3$, $h \in L^p(0, T; L^q(\Omega))$, and $f_0 \in V_0^{p,q}(\Omega)$. For this equation, we have

Lemma 4.1. *Assume that Ω is a bounded domain in \mathbb{R}^3 with $C^{2+\beta}$ ($\beta > 0$) boundary. Then there is a positive constant C , independent of T , such that*

$$\|f\|_{\mathcal{W}(0, T)} + \|\nabla h\|_{L^p(0, T; L^q(\Omega))} \leq C \left(\|g\|_{L^p(0, T; L^q(\Omega))} + \|f_0\|_{V_0^{p,q}(\Omega)} \right). \quad (4.2)$$

The strategy of proving Lemma 4.1 is to decompose the function f into two parts: one is a solution to Stokes equations and the other is a solution to a parabolic equation. For this purpose, let us first recall the maximal regularity theorem for parabolic equations (cf. [20]).

Proposition 4.1. *Assume that Ω is a bounded domain in \mathbb{R}^3 with $C^{2+\beta}$ ($\beta > 0$) boundary. Given $1 < p < \infty$, $\omega_0 \in V_0^{p,q}$ and $\varphi \in L^p(0, T; L^q(\Omega))$, the Cauchy problem*

$$\begin{cases} \frac{\partial \omega}{\partial t} - \mu \Delta \omega - (\mu + \lambda) \nabla \operatorname{div} \omega = \varphi, \\ \omega(0) = \omega_0, \end{cases}$$

has a unique solution $\omega \in \mathcal{W}(0, T)$, and

$$\|\omega\|_{\mathcal{W}(0, T)} \leq C_1 \left(\|\varphi\|_{L^p(0, T; L^q(\Omega))} + \|\omega_0\|_{V_0^{p,q}} \right),$$

where C_1 is independent of ω_0 and φ . In addition, there exists a positive constant c_0 independent of φ such that

$$\|\omega\|_{\mathcal{W}(0, T)} \geq c_0 \sup_{t \in (0, T)} \|\omega(t)\|_{V_0^{p,q}}.$$

The classical $L^p - L^q$ estimates for Stokes equations can be stated as follows (see Theorem 3.2 in [3]).

Proposition 4.2. *Assume that Ω is a bounded domain in \mathbb{R}^3 with $C^{2+\beta}$ ($\beta > 0$) boundary, and $\psi \in L^s(\mathbb{R}^+, L^q(\Omega))$ for $1 < q, s < \infty$. Then the initial-boundary value problem of the Stokes equations*

$$\begin{cases} \partial_t \mathbf{u} - \mu \Delta \mathbf{u} + \nabla \pi = \psi, & \int_{\Omega} \pi dx = 0, \\ \operatorname{div} \mathbf{u} = 0, \\ \mathbf{u}|_{\partial\Omega} = \mathbf{u}|_{t=0} = 0, \end{cases}$$

has a unique solution (\mathbf{u}, π) satisfying the following inequality for all $T > 0$:

$$\|(\Delta \mathbf{u}, \nabla \pi, \mathbf{u}_t)\|_{L^s(0, T; L^q(\Omega))} \leq C \|\psi\|_{L^s(0, T; L^q(\Omega))}$$

with $C = C(q, s, \Omega, \mu)$.

Proof of Lemma 4.1. To begin with, we first introduce two new functions f_1 and f_2 , which satisfy respectively

$$\begin{cases} \partial_t f_1 - \mu \Delta f_1 + \nabla h = g \\ \operatorname{div} f_1 = 0 \\ f_1(0) = 0, \quad f_1|_{\partial\Omega} = 0, \end{cases} \quad (4.3)$$

and

$$\begin{cases} \partial_t f_2 - \mu \Delta f_2 - (\mu + \lambda) \nabla \operatorname{div} f_2 = 0 \\ f_2(0) = f_0, \quad f_2|_{\partial\Omega} = 0. \end{cases} \quad (4.4)$$

Notice that

$$f = f_1 + f_2$$

by the linearity of the equation (4.1). For Stokes system (4.3), using Proposition 4.2, there exists a solution f_1 to (4.3) and it satisfies

$$\|f_1\|_{\mathcal{W}(0, T)} + \|\nabla h\|_{L^p(0, T; L^q(\Omega))} \leq C \|g\|_{L^p(0, T; L^q(\Omega))},$$

where the positive constant C depends on p, q, Ω and does not depend on T . On the other hand, by the maximal regularity theorem of parabolic equations, there exists a solution f_2 (extended to zero outside Ω) to (4.4) and f_2 satisfies

$$\|f_2\|_{\mathcal{W}(0, T)} \leq C \|f_0\|_{V_0^{p,q}},$$

where the positive constant C depends on μ, λ, p, q and does not depend on T . Hence, we have

$$\begin{aligned} \|f\|_{\mathcal{W}(0,T)} + \|\nabla h\|_{L^p(0,T;L^q(\Omega))} &\leq \|f_1\|_{\mathcal{W}(0,T)} + \|f_2\|_{\mathcal{W}(0,T)} + \|\nabla h\|_{L^p(0,T;L^q(\Omega))} \\ &\leq C \left(\|g\|_{L^p(0,T;L^q(\Omega))} + \|f_0\|_{V_0^{p,q}} \right), \end{aligned}$$

where the positive constant C depends on p, q, Ω and does not depend on T . \square

5. UNIFORM ESTIMATES

Due to Theorem 3.1, for any given initial data, the local existence and uniqueness of strong solution can be established. In order to extend the local solution to a global one, we need to establish a series of uniform bounds. That is the main objective of this section.

Throughout this section, we assume that over the time interval $[0, T]$, for a given sufficiently small positive number $R < 1$, the following bounds hold

$$\begin{cases} \|\mathbf{u}\|_{\mathcal{W}(0,T)} \leq R; \\ \|\rho\|_{L^\infty(0,T;W^{1,q}(\Omega))} \leq R; \\ \|E\|_{L^\infty(0,T;W^{1,q}(\Omega))} \leq R. \end{cases} \quad (5.1)$$

The main goal of this section is to obtain some uniform bounds on $\nabla \rho$ and ∇E in $L^p(0, T; L^q(\Omega))$ (see Corollary 5.1 below).

To begin with, notice that since $q > 3$, (5.1) will imply

$$\|\rho\|_{L^\infty(\Omega)} \leq C \|\rho\|_{W^{1,q}(\Omega)} \leq CR < \frac{1}{2},$$

if R is sufficiently small. Similarly, one can assume

$$\|E_{ij}\|_{L^\infty} \leq CR \leq 1, \quad \text{for all } i, j = 1, 2, 3.$$

5.1. Dissipation of the gradient of the density. To prove Theorem 3.2 valid, we need first to establish a uniform estimate on the dissipation of the gradient of the density.

Lemma 5.1. *Under the same condition as Theorem 3.2, the solution (ρ, \mathbf{u}, E) satisfies*

$$\|\nabla \rho\|_{L^p(0,T;L^q(\Omega))} \leq C(R^2 + R\|\nabla E\|_{L^p(0,T;L^q(\Omega))}), \quad (5.2)$$

where $C = C(p, q, \mu, \lambda, \Omega)$.

Proof. To begin with, we introduce the zero-th order differential operator

$$\mathcal{R}_{ij} = \Delta^{-1} \partial_{x_j} \partial_{x_i},$$

which is defined as

$$\mathcal{R}_{ij} f = -\mathcal{F}^{-1} \left(\frac{\xi_i \xi_j}{|\xi|^2} \mathcal{F} f \right),$$

where \mathcal{F} denotes the Fourier transformation. Note that

$$\|\mathcal{R}_{ij} f\|_{L^q} \leq C \|f\|_{L^q}, \quad \text{for } 1 < q < \infty \quad \text{and} \quad 1 \leq i, j \leq 3.$$

Extending the perturbations of the density and the deformation gradient by 0 outside the domain Ω , applying the operator \mathcal{R}_{ij} to (1.1b), and using the equation (1.1a), we obtain

$$\begin{aligned} & \mathcal{R}_{ij} ((1 + \rho)\mathbf{u}_t + (1 + \rho)\mathbf{u} \cdot \nabla \mathbf{u}) - (2\mu + \lambda)\partial_{x_j} \operatorname{div} \mathbf{u} + \alpha \partial_{x_j} \rho \\ &= \mathcal{R}_{ij} (\rho \operatorname{tr} E + (1 + \rho)E_{lk} \partial_{x_l} E_{ik}) + \mathcal{M}_j + (P'(1) - P'(1 + \rho)) \partial_{x_j} \rho, \end{aligned} \quad (5.3)$$

where

$$\alpha = (1 + P'(1)),$$

and we used the fact that, from (2.4) and (2.6),

$$\begin{aligned} \mathcal{R}_{ij} \operatorname{div} E &= \Delta^{-1} \partial_{x_j} \partial_{x_i} \partial_{x_k} E_{ik} = \Delta^{-1} \partial_{x_j} \partial_{x_k} \partial_{x_i} E_{ik} \\ &= \Delta^{-1} \partial_{x_j} \partial_{x_k} \partial_{x_k} E_{ii} + \mathcal{R}_{jk} (E_{li} \nabla_l E_{ik} - E_{lk} \nabla_l E_{ii}) \\ &= \partial_{x_j} \operatorname{tr} E + \mathcal{R}_{jk} (E_{li} \nabla_l E_{ik} - E_{lk} \nabla_l E_{ii}) \\ &= -\partial_{x_j} \rho - \partial_{x_j} (\rho \operatorname{tr} E) + \partial_{x_j} \left((1 + \rho) \left[\frac{1}{2} [\operatorname{tr}(E^2) - (\operatorname{tr} E)^2] - \det E \right] \right) \\ &\quad + \mathcal{R}_{jk} (E_{li} \nabla_l E_{ik} - E_{lk} \nabla_l E_{ii}) \\ &=: -\partial_{x_j} \rho + \mathcal{M}_j, \end{aligned} \quad (5.4)$$

with

$$\mathcal{M}_j = \partial_{x_j} \left((1 + \rho) \left[\frac{1}{2} [\operatorname{tr}(E^2) - (\operatorname{tr} E)^2] - \det E \right] \right) + \mathcal{R}_{jk} (E_{li} \nabla_l E_{ik} - E_{lk} \nabla_l E_{ii}).$$

We rewrite the equation (5.3) as

$$\begin{aligned} & (\mathcal{R}_{ij} \mathbf{u})_t - \mu \Delta \mathcal{R}_{ij} \mathbf{u} - (\mu + \lambda) \nabla \operatorname{div} \mathcal{R}_{ij} \mathbf{u} + \alpha \partial_{x_j} \rho \\ &= -\mathcal{R}_{ij} (\rho \mathbf{u}_t + (1 + \rho)\mathbf{u} \cdot \nabla \mathbf{u}) + \mathcal{R}_{ij} (\rho \operatorname{tr} E + (1 + \rho)E_{lk} \partial_{x_l} E_{ik}) + \mathcal{M}_j \\ &\quad + (P'(1) - P'(1 + \rho)) \partial_{x_j} \rho. \end{aligned} \quad (5.5)$$

According to Lemma 4.1, we have

$$\begin{aligned} & \|\mathcal{R}_{ij} \mathbf{u}\|_{\mathcal{W}(0,T)} + \alpha \|\nabla \rho\|_{L^p(0,T;L^q(\Omega))} \\ & \leq C \left(\|\mathcal{R}_{ij} \mathbf{u}_0\|_{V_0^{p,q}} + \|\mathcal{R}_{ij} (\rho \mathbf{u}_t + (1 + \rho)\mathbf{u} \cdot \nabla \mathbf{u})\|_{L^p(0,T;L^q(\Omega))} \right. \\ & \quad + \|\mathcal{R}_{ij} (\rho \operatorname{tr} E + (1 + \rho)E_{lk} \partial_{x_l} E_{ik})\|_{L^p(0,T;L^q(\Omega))} \\ & \quad \left. + \|\mathcal{M}_j\|_{L^p(0,T;L^q(\Omega))} + \|(P'(1) - P'(1 + \rho)) \partial_{x_j} \rho\|_{L^p(0,T;L^q(\Omega))} \right). \end{aligned} \quad (5.6)$$

Next, we estimate the terms on the right hand side of the above inequality. Indeed,

$$\begin{aligned} & \|\mathcal{R}_{ij} (\rho \mathbf{u}_t + (1 + \rho)\mathbf{u} \cdot \nabla \mathbf{u})\|_{L^p(0,T;L^q(\Omega))} \\ & \leq \|\rho \mathbf{u}_t + (1 + \rho)\mathbf{u} \cdot \nabla \mathbf{u}\|_{L^p(0,T;L^q(\Omega))} \\ & \leq \|\rho\|_{L^\infty} \|\mathbf{u}_t\|_{L^p(0,T;L^q(\Omega))} + (1 + \|\rho\|_\infty) \|\mathbf{u}\|_{L^\infty(0,T;L^q(\Omega))} \|\nabla \mathbf{u}\|_{L^p(0,T;L^\infty(\Omega))} \\ & \leq C \left(\|\rho\|_{L^\infty} \|\mathbf{u}_t\|_{L^p(0,T;L^q(\Omega))} + (1 + \|\rho\|_\infty) \|\mathbf{u}\|_{L^\infty(0,T;L^q(\Omega))} \|\mathbf{u}\|_{L^p(0,T;W^{2,q}(\Omega))} \right) \\ & \leq CR^2; \end{aligned}$$

$$\begin{aligned}
& \|\mathcal{R}_{ij}(\rho \operatorname{tr} E + (1 + \rho) E_{lk} \partial_{x_l} E_{ik})\|_{L^p(0,T;L^q(\Omega))} \\
& \leq \|\rho \operatorname{tr} E + (1 + \rho) E_{lk} \partial_{x_l} E_{ik}\|_{L^p(0,T;L^q(\Omega))} \\
& \leq \|E\|_{L^\infty} \|\rho\|_{L^p(0,T;L^q(\Omega))} + (1 + \|\rho\|_{L^\infty}) \|E\|_{L^\infty} \|\nabla E\|_{L^p(0,T;L^q(\Omega))} \\
& \leq C(R^2 + R \|\nabla E\|_{L^p(0,T;L^q(\Omega))});
\end{aligned}$$

$$\begin{aligned}
\|\mathcal{M}_j\|_{L^p(0,T;L^q(\Omega))} & \leq \|\mathcal{R}(E_{li} \nabla_l E_{ik} - E_{lk} \nabla_l E_{ii})\|_{L^p(0,T;L^q(\Omega))} \\
& \quad + \left\| \partial_{x_j} \left((1 + \rho) \left[\frac{1}{2} [\operatorname{tr}(E^2) - (\operatorname{tr} E)^2] - \det E \right] \right) \right\|_{L^p(0,T;L^q(\Omega))} \\
& \leq C \left(\|E_{li} \nabla_l E_{ik} - E_{lk} \nabla_l E_{ii}\|_{L^p(0,T;L^q(\Omega))} \right. \\
& \quad + \|\nabla \rho\|_{L^p(0,T;L^q(\Omega))} (\|E\|_{L^\infty}^2 + \|E\|_{L^\infty}^3) \\
& \quad + (1 + \|\rho\|_{L^\infty}) \|\nabla E\|_{L^p(0,T;L^q(\Omega))} (\|E\|_{L^\infty} + \|E\|_{L^\infty}^2) \Big) \\
& \leq C \left(R \|\nabla E\|_{L^p(0,T;L^q(\Omega))} + R \|\nabla \rho\|_{L^p(0,T;L^q(\Omega))} \right),
\end{aligned} \tag{5.7}$$

since $R < 1$;

$$\begin{aligned}
\|(P'(1) - P'(1 + \rho)) \partial_{x_j} \rho\|_{L^p(0,T;L^q(\Omega))} & \leq \eta \|\partial_{x_j} \rho\|_{L^p(0,T;L^q(\Omega))} \|\rho\|_{L^\infty} \\
& \leq CR \|\nabla \rho\|_{L^p(0,T;L^q(\Omega))},
\end{aligned}$$

here we used the fact

$$P'(1 + \rho) - P'(1) = P''(z) \rho \quad \text{for some } z \text{ between } 1 \text{ and } 1 + \rho,$$

and hence

$$\|P'(1 + \rho) - P'(1)\|_{L^\infty} \leq \eta \|\rho\|_{L^\infty}$$

with

$$\eta = \sup_{\frac{1}{2} \leq z \leq \frac{3}{2}} |P''(z)|.$$

Substituting those estimates back into (5.6), we obtain

$$\begin{aligned}
& \|\mathcal{R}_{ij} \mathbf{u}\|_{\mathcal{W}(0,T)} + \alpha \|\nabla \rho\|_{L^p(0,T;L^q(\Omega))} \\
& \leq C \left(R^2 + R \|\nabla E\|_{L^p(0,T;L^q(\Omega))} + R \|\nabla \rho\|_{L^p(0,T;L^q(\Omega))} \right).
\end{aligned}$$

Assuming that R is so small that $CR < \frac{\alpha}{2}$, we deduce from the above inequality that

$$\|\mathcal{R}_{ij} \mathbf{u}\|_{\mathcal{W}(0,T)} + \frac{\alpha}{2} \|\nabla \rho\|_{L^p(0,T;L^q(\Omega))} \leq C \left(R^2 + R \|\nabla E\|_{L^p(0,T;L^q(\Omega))} \right). \tag{5.8}$$

The proof is complete. \square

5.2. Dissipation of the deformation gradient. The main difficulty of the proof of Theorem 3.2 is to obtain estimates on the dissipation of the deformation gradient. This is partly because of the transport structure of equation (1.3c). It is worthy of pointing out that it is extremely difficult to directly deduce the dissipation of the deformation gradient. Fortunately, for the viscoelastic fluids system (1.3), as we can see in [2, 9, 10, 12, 13, 14], some sort of combinations between the gradient of the velocity and the deformation gradient indeed induce some dissipation. To make this statement more precise, we rewrite the momentum equation (1.3b) as, using (2.2)

$$\begin{aligned} \partial_t \mathbf{u} - \mu \Delta \mathbf{u} - \operatorname{div} E &= -(1 + \rho)(\mathbf{u} \cdot \nabla) \mathbf{u} - \nabla P(1 + \rho) \\ &\quad + \rho \operatorname{div} E + (1 + \rho) E_{jk} \partial_{x_j} E_{ik} - \rho \partial_t \mathbf{u}, \end{aligned} \quad (5.9)$$

and prove the following estimate:

Lemma 5.2. *Under the same condition as Theorem 3.2, the solution (ρ, \mathbf{u}, E) satisfies*

$$\|\nabla E\|_{L^p(0,T;L^q(\Omega))} \leq CR, \quad (5.10)$$

where $C = C(p, q, \mu, \lambda)$.

Proof. Now we introduce the function $Z_1(x, t) = \mathcal{L}(\operatorname{div} E)$ as

$$\begin{cases} -\mu \Delta \mathcal{L}(f) - (\mu + \lambda) \nabla \operatorname{div} \mathcal{L}(f) = f, \\ \mathcal{L}(f)|_{\partial\Omega} = 0. \end{cases} \quad (5.11)$$

Then, (5.9) becomes

$$\partial_t \mathbf{u} - \mu \Delta \left(\mathbf{u} - \frac{1}{\mu} Z_1 \right) = \mathcal{H}_1, \quad (5.12)$$

where

$$\mathcal{H}_1 = -(1 + \rho)(\mathbf{u} \cdot \nabla) \mathbf{u} - \nabla P(1 + \rho) + \rho \operatorname{div} E + (1 + \rho) E_{jk} \partial_{x_j} E_{ik} - \rho \partial_t \mathbf{u}.$$

Also, from (1.3c), we have

$$\begin{aligned} \frac{\partial Z_1}{\partial t} &= \mathcal{L} \left(\operatorname{div} \frac{\partial E}{\partial t} \right) \\ &= \mathcal{L} \left(\operatorname{div} (\nabla \mathbf{u} + \nabla \mathbf{u} E - (\mathbf{u} \cdot \nabla) E) \right). \end{aligned} \quad (5.13)$$

From (5.12) and (5.13), we deduce, denoting $Z = \mathbf{u} - \frac{1}{\mu} Z_1$,

$$\partial_t Z - \mu \Delta Z - (\mu + \lambda) \nabla \operatorname{div} Z = \mathcal{H} := \mathcal{H}_1 - \mathcal{H}_2, \quad (5.14)$$

where

$$\mathcal{H}_2 = \frac{1}{\mu} \mathcal{L}(\operatorname{div} (\nabla \mathbf{u} + \nabla \mathbf{u} E - (\mathbf{u} \cdot \nabla) E)).$$

Equation (5.14) with Proposition 4.1 implies that

$$\begin{aligned} \|Z\|_{\mathcal{W}(0,T)} &\leq C(p, q) \left(\|Z(0)\|_{X_p^{2(1-\frac{1}{p})}} + \|\mathcal{H}\|_{L^p(0,T;L^q(\Omega))} \right) \\ &\leq C(p, q) (R + \|\mathcal{H}\|_{L^p(0,T;L^q(\Omega))}). \end{aligned} \quad (5.15)$$

Next, we estimate $\|\mathcal{H}_i\|_{L^p(0,T;L^q(\Omega))}$, $i = 1, 2$. Indeed, for \mathcal{H}_1 , using (5.2), we have

$$\begin{aligned}
\|\mathcal{H}_1\|_{L^p(0,T;L^q(\Omega))} &\leq (1 + \|\rho\|_{L^\infty(Q_T)})\|\mathbf{u}\|_{L^\infty(0,T;L^q(\Omega))}\|\nabla\mathbf{u}\|_{L^p(0,T;L^\infty(\Omega))} \\
&\quad + \kappa\|\nabla\rho\|_{L^p(0,T;L^q(\Omega))} + \|\rho\|_{L^\infty(Q_T)}\|\nabla E\|_{L^p(0,T;L^q(\Omega))} \\
&\quad + (1 + \|\rho\|_{L^\infty(Q_T)})\|E\|_{L^\infty(Q_T)}\|\nabla E\|_{L^p(0,T;L^q(\Omega))} \\
&\quad + \|\rho\|_{L^\infty(Q_T)}\|\partial_t\mathbf{u}\|_{L^p(0,T;L^q(\Omega))} \\
&\leq C\left(R + R\|\mathbf{u}\|_{L^p(0,T;W^{2,q}(\Omega))} + R\|\nabla E\|_{L^p(0,T;L^q(\Omega))}\right) \\
&\leq C\left(R + R\|\nabla E\|_{L^p(0,T;L^q(\Omega))}\right),
\end{aligned} \tag{5.16}$$

since $R < 1$ with

$$\kappa = \sup_{|z| \leq \frac{1}{2}} P'(1+z).$$

For \mathcal{H}_2 , we have

$$|\mathcal{H}_2| \leq \frac{1}{\mu}|\mathbf{u}| + \frac{1}{\mu}\mathcal{L}(\operatorname{div}(\nabla\mathbf{u}E - (\mathbf{u} \cdot \nabla)E)),$$

and

$$\begin{aligned}
\|\nabla\mathbf{u}E - (\mathbf{u} \cdot \nabla)E\|_{L^p(0,T;L^q(\Omega))} &\leq \|\nabla\mathbf{u}\|_{L^p(0,T;L^q(\Omega))}\|E\|_{L^\infty} \\
&\quad + \|\mathbf{u}\|_{L^\infty}\|\nabla E\|_{L^p(0,T;L^q(\Omega))} \\
&\leq R^2.
\end{aligned}$$

Hence, one can estimate, by the standard estimates of elliptic equations,

$$\begin{aligned}
\|\mathcal{H}_2\|_{L^p(0,T;L^q(\Omega))} &\leq \frac{1}{\mu}\|\mathbf{u}\|_{L^p(0,T;L^q(\Omega))} + \frac{1}{\mu}\|\nabla\mathbf{u}E - (\mathbf{u} \cdot \nabla)E\|_{L^p(0,T;L^q(\Omega))} \\
&\leq C(R + R^2) \leq CR.
\end{aligned} \tag{5.17}$$

Therefore, from (5.16) and (5.17), we obtain

$$\|\mathcal{H}\|_{L^p(0,T;L^q(\Omega))} \leq C\left(R + R\|\nabla E\|_{L^p(0,T;L^q(\Omega))}\right). \tag{5.18}$$

Inequalities (5.15) and (5.18) imply that

$$\|Z\|_{L^p(0,T;W^{2,q}(\Omega))} \leq C\left(R + R\|\nabla E\|_{L^p(0,T;L^q(\Omega))}\right). \tag{5.19}$$

Hence, we have, from (5.11)

$$\begin{aligned}
\|\operatorname{div}E\|_{L^p(0,T;L^q(\Omega))} &\leq \mu\left(\|Z\|_{L^p(0,T;W^{2,q}(\Omega))} + \|\mathbf{u}\|_{L^p(0,T;W^{2,q}(\Omega))}\right) \\
&\leq C\left(R + R\|\nabla E\|_{L^p(0,T;L^q(\Omega))}\right).
\end{aligned} \tag{5.20}$$

On the other hand, from the identity (2.4), we deduce that

$$\begin{aligned}
\|\operatorname{curl} E_i\|_{L^p(0,T;L^q(\Omega))} &\leq 2\|E\|_{L^\infty(Q_T)}\|\nabla E\|_{L^p(0,T;L^q(\Omega))} \\
&\leq C\|E\|_{L^\infty(0,T;W^{1,q}(\Omega))}\|\nabla E\|_{L^p(0,T;L^q(\Omega))} \\
&\leq CR\|\nabla E\|_{L^p(0,T;L^q(\Omega))}.
\end{aligned} \tag{5.21}$$

Combining together (5.20), (5.21) and the inequality

$$\|\nabla f\|_{L^r} \leq C(\|\operatorname{div}f\|_{L^r} + \|\operatorname{curl}f\|_{L^r}) \text{ for all } 1 < r < \infty,$$

we obtain

$$\|\nabla E\|_{L^p(0,T;L^q(\Omega))} \leq C \left(R + R \|\nabla E\|_{L^p(0,T;L^q(\Omega))} \right),$$

and hence, by choosing $CR \leq \frac{1}{2}$, one obtains (5.10).

The proof of Lemma 5.2 is complete. \square

Notice that, from (5.2) and (5.10), one can improve the estimates on the gradient of the density and the divergence of the deformation gradient. Indeed, we have

Corollary 5.1. *Under the same condition as Theorem 3.2, the solution (ρ, \mathbf{u}, E) satisfies*

$$\|\nabla \rho\|_{L^p(0,T;L^q(\Omega))} \leq CR^2, \quad (5.22)$$

and

$$\|\nabla E\|_{L^p(0,T;L^q(\Omega))} \leq CR^2, \quad (5.23)$$

where the positive constant C is independent of T .

Proof. First, note that the estimate (5.22) is a direct consequence of Lemma 5.1 and Lemma 5.2.

To show (5.23), according to (5.21) and Lemma 5.2, we only need to verify

$$\|\operatorname{div} E\|_{L^p(0,T;L^q(\Omega))} \leq CR^2.$$

To this end, we notice first that, since $\operatorname{div} \operatorname{curl} f = 0$ for any vector-valued function f , thus

$$(-\Delta)^{-1} \operatorname{div} \operatorname{curl} \operatorname{div} E = 0.$$

On the other hand, we also have, using the identity (2.4),

$$\begin{aligned} (-\Delta)^{-1} \operatorname{div} \operatorname{curl} \operatorname{div} E &= (-\Delta)^{-1} \operatorname{div} (\partial_{x_k} \partial_{x_j} E_{ij} - \partial_{x_i} \partial_{x_j} E_{kj}) \\ &= (-\Delta)^{-1} \operatorname{div} (\partial_{x_j} \partial_{x_j} E_{ik} - \partial_{x_j} \partial_{x_j} E_{ki}) \\ &\quad + (-\Delta)^{-1} \operatorname{div} \partial_{x_j} (E_{lj} \nabla_l E_{ik} - E_{lk} \nabla_l E_{ij} - E_{lj} \nabla_l E_{ki} + E_{li} \nabla_l E_{kj}) \\ &= \operatorname{div} E - \operatorname{div} E^\top + \mathcal{N} \end{aligned}$$

with

$$\mathcal{N} = (-\Delta)^{-1} \operatorname{div} \partial_{x_j} (E_{lj} \nabla_l E_{ik} - E_{lk} \nabla_l E_{ij} - E_{lj} \nabla_l E_{ki} + E_{li} \nabla_l E_{kj}).$$

Hence, we have

$$\operatorname{div} E = \operatorname{div} E^\top - \mathcal{N}. \quad (5.24)$$

The property (2.2) implies that

$$\operatorname{div} E^\top = -\nabla \rho - \operatorname{div}(\rho E^\top),$$

which together with (5.22) yield

$$\begin{aligned} \|\operatorname{div} E^\top\|_{L^p(0,T;L^q(\Omega))} &= \|\nabla \rho + \operatorname{div}(\rho E^\top)\|_{L^p(0,T;L^q(\Omega))} \\ &\leq \|\nabla \rho\|_{L^p(0,T;L^q(\Omega))} + \|\rho\|_{L^\infty} \|\nabla E\|_{L^p(0,T;L^q(\Omega))} \\ &\quad + \|E\|_{L^\infty} \|\nabla \rho\|_{L^p(0,T;L^q(\Omega))} \\ &\leq CR^2. \end{aligned} \quad (5.25)$$

For \mathcal{N} , we have, using Lemma 5.2,

$$\begin{aligned}\|\mathcal{N}\|_{L^p(0,T;L^q(\Omega))} &\leq 4\|E\|_{L^\infty}\|\nabla E\|_{L^p(0,T;L^q(\Omega))} \\ &\leq CR^2\end{aligned}\tag{5.26}$$

Combining (5.24), (5.25), and (5.26), we have

$$\|\operatorname{div} E\|_{L^p(0,T;L^q(\Omega))} \leq CR^2.$$

The proof is complete. \square

6. PROOF OF THEOREM 3.2

The goal of this section is to prove Theorem 3.2. To this end, for a given $R > 0$ small enough, we define

$$\begin{aligned}T_{\max} &:= \sup \left\{ T > 0 : \text{there exists a solution } (\rho, \mathbf{u}, E) \text{ to (1.3) with } \|\mathbf{u}\|_{\mathcal{W}(0,T)} < R \text{ and} \right. \\ &\quad \left. \max\{\|\rho\|_{L^\infty(0,T;W^{1,q}(\Omega))}, \|E\|_{L^\infty(0,T;W^{1,q}(\Omega))}\} < R \right\}.\end{aligned}$$

We first show that in the interval $[0, T_{\max}]$, we have better estimates on the velocity.

Lemma 6.1. *For any $T \in [0, T_{\max}]$, we have*

$$\|\mathbf{u}\|_{\mathcal{W}(0,T)} \leq CR^2$$

for some positive constant C independent of T .

Proof. In fact, by Proposition 4.1 and equation (1.3b), we have

$$\begin{aligned}\|\mathbf{u}\|_{\mathcal{W}(0,T)} &\leq C(p, q, \Omega) \left(\|\mathbf{u}_0\|_{V_0^{p,q}} + \|\rho \partial_t \mathbf{u}\|_{L^p(0,T;L^q(\Omega))} \right. \\ &\quad + \|(1 + \rho) \mathbf{u} \cdot \nabla \mathbf{u}\|_{L^p(0,T;L^q(\Omega))} \\ &\quad + \|\nabla P(1 + \rho)\|_{L^p(0,T;L^q(\Omega))} \\ &\quad + \|(1 + \rho) \operatorname{div} E\|_{L^p(0,T;L^q(\Omega))} \\ &\quad \left. + \|(1 + \rho) E_{jk} \partial_{x_j} E_{ik}\|_{L^p(0,T;L^q(\Omega))} \right).\end{aligned}\tag{6.1}$$

For those terms on the right hand side of (6.1), in view of Lemma 5.2 and Corollary 5.1, we have the following estimates:

$$\begin{aligned}\|\rho \partial_t \mathbf{u}\|_{L^p(0,T;L^q(\Omega))} &\leq \|\rho\|_{L^\infty} \|\partial_t \mathbf{u}\|_{L^p(0,T;L^q(\Omega))} \leq CR^2; \\ \|(1 + \rho) \mathbf{u} \cdot \nabla \mathbf{u}\|_{L^p(0,T;L^q(\Omega))} &\leq (1 + \|\rho\|_{L^\infty}) \|\mathbf{u}\|_{L^\infty(0,T;L^q(\Omega))} \|\nabla \mathbf{u}\|_{L^p(0,T;L^\infty)} \\ &\leq C \|\mathbf{u}\|_{L^\infty(0,T;L^q(\Omega))} \|\mathbf{u}\|_{L^p(0,T;W^{2,q}(\Omega))} \\ &\leq CR^2; \\ \|\nabla P(1 + \rho)\|_{L^p(0,T;L^q(\Omega))} &\leq \kappa \|\nabla \rho\|_{L^p(0,T;L^q(\Omega))} \leq CR^2; \\ \|(1 + \rho) \operatorname{div} E\|_{L^p(0,T;L^q(\Omega))} &\leq (1 + \|\rho\|_{L^\infty}) \|\operatorname{div} E\|_{L^p(0,T;L^q(\Omega))} \leq CR^2; \\ \|(1 + \rho) E_{jk} \partial_{x_j} E_{ik}\|_{L^p(0,T;L^q(\Omega))} &\leq (1 + \|\rho\|_{L^\infty}) \|E\|_{L^\infty} \|\nabla E\|_{L^p(0,T;L^q(\Omega))} \\ &\leq C \|E\|_{L^\infty} \|\nabla E\|_{L^p(0,T;L^q(\Omega))} \\ &\leq CR^2.\end{aligned}$$

Substituting those estimates back into (6.1), we have

$$\|\mathbf{u}\|_{\mathcal{W}(0,T)} \leq C(\|\mathbf{u}_0\|_{V_0^{p,q}} + R^2) \leq CR^2,$$

provided that we choose the initial data to satisfy

$$\|\mathbf{u}_0\|_{V_0^{p,q}} \leq CR^2.$$

The proof is complete. □

According to Lemma 6.1, we deduce from the above estimate that

$$\|\mathbf{u}\|_{\mathcal{W}(0,T_{\max})} \leq CR^2.$$

Next, we improve the estimates on the density and the deformation gradient. Due to the similarity of equations for the density and the deformation gradient, the arguments for those two unknowns are similar, and hence for the clarity of the presentation, we will only focus on the argument for the improved estimates for the density. To this end, we introduce a new variable:

$$\sigma := \nabla \ln \varrho,$$

and we have

Lemma 6.2. *Function σ satisfies*

$$\partial_t \sigma + \nabla(\mathbf{u} \cdot \sigma) = -\nabla \operatorname{div} \mathbf{u}, \quad (6.2)$$

in the sense of distributions. Moreover, the norm $\|\sigma(t)\|_{L^q(\Omega)}$ is continuous in time.

Proof. We follow the argument in [20] (Section 9.8) by denoting $\sigma_\varepsilon = S_\varepsilon \sigma$, where S_ε is the standard mollifier in the spatial variables. Then, we have

$$\partial_t \sigma_\varepsilon + \nabla(\mathbf{u} \cdot \sigma_\varepsilon) = -\nabla \operatorname{div} \mathbf{u}_\varepsilon + \mathcal{R}_\varepsilon,$$

with

$$\begin{aligned} \mathcal{R}_\varepsilon &= \nabla(\mathbf{u} \cdot \sigma_\varepsilon) - S_\varepsilon \nabla(\mathbf{u} \cdot \sigma) \\ &= (\mathbf{u} \cdot \nabla \sigma_\varepsilon - S_\varepsilon(\mathbf{u} \cdot \nabla \sigma)) + (\sigma_\varepsilon \nabla \mathbf{u} - S_\varepsilon(\sigma \cdot \nabla \mathbf{u})) \\ &=: \mathcal{R}_\varepsilon^1 + \mathcal{R}_\varepsilon^2. \end{aligned} \quad (6.3)$$

Since $\sigma \in L^\infty(0, T; L^q(\Omega))$ and $\mathbf{u} \in L^p(0, T; W^{1,\infty}(\Omega))$, we deduce from Lemma 6.7 in [20] (cf. Lemma 2.3 in [15]) that

$$\mathcal{R}_\varepsilon^1 \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0.$$

Moreover,

$$\|(\sigma_\varepsilon - \sigma) \nabla \mathbf{u}\|_{L^1(0,T;L^q(\Omega))} \leq \|\sigma - \sigma_\varepsilon\|_{L^{\frac{p}{p-1}}(0,T;L^q(\Omega))} \|\nabla \mathbf{u}\|_{L^p(0,T;L^\infty(\Omega))} \rightarrow 0,$$

and

$$S_\varepsilon(\sigma \cdot \nabla \mathbf{u}) \rightarrow \sigma \cdot \nabla \mathbf{u} \quad \text{in } L^1(0, T; L^q(\Omega))$$

since $\sigma \cdot \nabla \mathbf{u} \in L^p(0, T; L^q(\Omega))$. Thus, we have

$$\mathcal{R}_\varepsilon^2 \rightarrow 0 \quad \text{in } L^p(0, T; L^q(\Omega)).$$

Then, taking the limit as $\varepsilon \rightarrow 0$ in (6.3), we get (6.2).

Multiplying (6.2) by $|\sigma|^{q-2}\sigma$, and integrating over Ω , we get

$$\begin{aligned} \left| \frac{1}{q} \frac{d}{dt} \|\sigma\|_{L^q(\Omega)}^q \right| &= \left| \int_{\Omega} (-\nabla \operatorname{div} \mathbf{u} \cdot \sigma |\sigma|^{q-2} - \partial_j \mathbf{u}_k \sigma_j \sigma_k |\sigma|^{q-2} - \frac{1}{q} \operatorname{div} \mathbf{u} |\sigma|^q) dx \right| \\ &\leq \|\mathbf{u}\|_{W^{2,q}(\Omega)} \|\sigma\|_{L^q(\Omega)}^{q-1} + \|\nabla \mathbf{u}\|_{L^\infty} \|\sigma\|_{L^q}^q + \frac{1}{q} \|\operatorname{div} \mathbf{u}\|_{L^\infty} \|\sigma\|_{L^q}^q \\ &\leq C \|\mathbf{u}\|_{W^{2,q}} \|\sigma\|_{L^q}^{q-1} (1 + \|\sigma\|_{L^q}). \end{aligned}$$

Dividing the above inequality by $\|\sigma\|_{L^q}^{q-1}$, we obtain

$$\left| \frac{d}{dt} \|\sigma\|_{L^q} \right| \leq C \|\mathbf{u}\|_{W^{2,q}} (1 + \|\sigma\|_{L^q}).$$

Since $\sigma \in L^\infty(0, T; L^q(\Omega))$, we have

$$\frac{d}{dt} \|\sigma\|_{L^q} \in L^p(0, T).$$

Thus, $\|\sigma\|_{L^q} \in C(0, T)$. The proof of Lemma 6.2 is complete. \square

With the aid of Lemma 6.2, we can improve the estimates on the density.

Lemma 6.3. *For any $T \in [0, T_{max}]$, the density ϱ satisfies*

$$\|\nabla \varrho\|_{L^\infty(0, T; L^q(\Omega))} \leq C R^{\frac{3}{2}}$$

where the positive constant C is independent of T .

Proof. We only need to prove

$$\|\sigma\|_{L^\infty(0, T; L^q(\Omega))} \leq C R^{\frac{3}{2}}.$$

Indeed, multiplying (6.2) by $|\sigma|^{q-2}\sigma$ and integrating over Ω yields

$$\frac{1}{q} \frac{d}{dt} \|\sigma(t)\|_{L^q(\Omega)}^q = - \int_{\Omega} \nabla \operatorname{div} \mathbf{u} \cdot \sigma |\sigma|^{q-2} - \int_{\Omega} \nabla(\mathbf{u} \cdot \sigma) \cdot \sigma |\sigma|^{q-2} dx. \quad (6.4)$$

Note that

$$\left| \int_{\Omega} \nabla \operatorname{div} \mathbf{u} \cdot \sigma |\sigma|^{q-2} \right| \leq \|\Delta \mathbf{u}\|_{L^q(\Omega)} \|\sigma\|_{L^q(\Omega)}^{q-1};$$

and

$$\int_{\Omega} \nabla(\mathbf{u} \cdot \sigma) \cdot \sigma |\sigma|^{q-2} dx = \int_{\Omega} \partial_{x_j} \mathbf{u}_k \sigma_k \sigma_j |\sigma|^{q-2} dx + I$$

where

$$\begin{aligned} I &=: \int_{\Omega} \partial_{x_k} \partial_{x_j} \partial_{x_k} (\ln \varrho) \partial_{x_j} \ln \varrho |\sigma|^{q-2} dx \\ &= \frac{1}{2} \int_{\Omega} \mathbf{u}_k \partial_{x_k} \left(\sum_{j=1}^3 \sigma_j^2 \right) |\sigma|^{q-2} dx = \int_{\Omega} \mathbf{u}_k (\partial_{x_k} |\sigma|) |\sigma|^{q-1} dx \\ &= \frac{1}{q} \int_{\Omega} \mathbf{u}_k \partial_{x_k} (|\sigma|^q) dx = -\frac{1}{q} \int_{\Omega} |\sigma|^q \operatorname{div} \mathbf{u} dx. \end{aligned}$$

Hence,

$$\left| \int_{\Omega} \nabla(\mathbf{u} \cdot \sigma) \cdot \sigma |\sigma|^{q-2} dx \right| \leq C \|\nabla \mathbf{u}\|_{L^\infty} \|\sigma\|_{L^q}^q \leq C \|\mathbf{u}\|_{W^{2,q}(\Omega)} \|\sigma\|_{L^q}^q.$$

Substituting those estimates back into (6.4), one obtains

$$\frac{1}{q} \frac{d}{dt} \|\sigma(t)\|_{L^q(\Omega)}^q \leq C \|\sigma\|_{L^q}^{q-1} \|\mathbf{u}\|_{W^{2,q}(\Omega)} (1 + \|\sigma\|_{L^q}).$$

Multiplying the above inequality by $\|\sigma\|_{L^q}^{p-q}$ and integrating over $[0, T]$, we have, using Corollary 5.1 and Lemma 6.1,

$$\begin{aligned} \frac{1}{q} \|\sigma(T)\|_{L^q}^p &\leq \frac{1}{q} \|\sigma_0\|_{L^q}^p + C \|\sigma\|_{L^p(0,T;L^q(\Omega))}^{p-1} \|\mathbf{u}\|_{L^p(0,T;W^{2,q}(\Omega))} (1 + \|\sigma\|_{L^\infty(0,T;L^q(\Omega))}) \\ &\leq \frac{1}{q} \|\sigma_0\|_{L^q}^p + CR^{2p} (1 + \|\sigma\|_{L^\infty(0,T;L^q(\Omega))}) \\ &\leq CR^{2p} (1 + \|\sigma\|_{L^\infty(0,T;L^q(\Omega))}). \end{aligned} \quad (6.5)$$

Thanks to Lemma 6.2, the continuity of $t \mapsto \|\sigma(t)\|_{L^q(\Omega)}$ implies that

$$\|\sigma(t)\|_{L^q(\Omega)} \leq R^{\frac{3}{2}}$$

in some maximal interval $(0, T(R)) \subset (0, T_{\max})$. If $T(R) < T_{\max}$, then we have

$$\|\sigma(T(R))\|_{L^q(\Omega)} = R^{\frac{3}{2}}.$$

On the other hand, (6.5) implies

$$\begin{aligned} R^{\frac{3}{2}} &= \|\sigma(T(R))\|_{L^q(\Omega)} \leq C^{\frac{1}{p}} R^2 (1 + \|\sigma\|_{L^\infty(0,T;L^q(\Omega))}) \\ &\leq C^{\frac{1}{p}} R^2 (1 + R^{\frac{3}{2}}). \end{aligned}$$

This is impossible, if we assume that R is so small that

$$C^{\frac{1}{p}} R^{\frac{1}{2}} (1 + R^{\frac{3}{2}}) < 1.$$

Thus, this implies that

$$\|\sigma\|_{L^\infty(0,T_{\max};L^q(\Omega))} \leq R^{\frac{3}{2}}.$$

The proof is complete. \square

Similarly, we have

$$\|\nabla E\|_{L^\infty(0,T_{\max};L^q(\Omega))} \leq CR^{\frac{3}{2}},$$

and hence

$$\max\{\|\nabla \rho\|_{L^\infty(0,T_{\max};L^q(\Omega))}, \|\nabla E\|_{L^\infty(0,T_{\max};L^q(\Omega))}\} \leq CR^{\frac{3}{2}}. \quad (6.6)$$

On the other hand, Proposition 2.2, combined with the initial conditions (3.2) and (3.3), imply that

$$\int_{\Omega} \rho dx = 0, \quad \text{and} \quad \int_{\Omega} (1 + \rho) E_{ij} = 0, \quad \text{for all } i, j = 1, 2, 3. \quad (6.7)$$

Poincare's inequality, combined with (6.6) and (6.7), yields

$$\max\{\|\rho\|_{L^\infty(0,T_{\max};L^q(\Omega))}, \|E\|_{L^\infty(0,T_{\max};L^q(\Omega))}\} \leq CR^{\frac{3}{2}},$$

and thus

$$\max\{\|\rho\|_{L^\infty(0,T_{\max};W^{1,q}(\Omega))}, \|E\|_{L^\infty(0,T_{\max};W^{1,q}(\Omega))}\} \leq CR^{\frac{3}{2}}. \quad (6.8)$$

Now we are in a position to finish the proof of Theorem 3.2.

Proof of Theorem 3.2. Suppose that $T_{\max} < \infty$. Let $T_n \nearrow T_{\max}$ be arbitrary. Then necessarily

$$\|\mathbf{u}\|_{\mathcal{W}(0,T_n)} \nearrow R \quad \text{as } n \rightarrow \infty.$$

Indeed, if this was not the case, then

$$\sup_n \|\mathbf{u}\|_{\mathcal{W}(0,T_n)} < R$$

and the function \mathbf{u} , defined on $(0, T_{\max})$, would be a solution to the equation (1.1) on $(0, T_{\max})$. Taking $\mathbf{u}(T_{\max})$ as a new initial condition, from Lemma 4.1, we get $\mathbf{u}(T_{\max}) \in V_0^{p,q}$. And then, according to Theorem 3.1, there exists a T_0 such that there exists a unique solution to (1.1) on $(T_{\max}, T_{\max} + T_0)$. This means we can extend the solution we obtained outside the interval $(0, T_{\max})$ to $(0, T_{\max} + T_0)$ with $\|\mathbf{u}\|_{\mathcal{W}(0, T_{\max} + T_0)} < R$ and

$$\max\{\|\rho\|_{L^\infty(0, T_{\max} + T_0; W^{1,q}(\Omega))}, \|E\|_{L^\infty(0, T_{\max} + T_0; W^{1,q}(\Omega))}\} < R$$

for some $T_0 > 0$. This is a contradiction with the maximality of T_{\max} . Thus, it is impossible to have

$$T_{\max} < \infty.$$

Hence, the solution is defined for all positive time.

Finally, from the estimate (6.8), one has

$$\|\rho\|_{L^\infty(0, \infty; W^{1,q}(\Omega))} < R$$

and

$$\|E\|_{L^\infty(0, \infty; W^{1,q}(\Omega))} < R.$$

The proof is complete. □

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